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The Nature of Mathematical Truth: A Comparative Study of Frege, Mill, and Kripke

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Abstract

Philosophers have debated the question of what makes mathematical statements true. In the present paper, we explore three influential and contrasting perspectives on mathematical truth by John Stuart Mill, Gottlob Frege and Saul Kripke. For the empiricist MILL, mathematics is inductive, its truths being generalisations of repeated observation. This perspective links mathematics directly to science, but it has been criticised for failing to account for the necessity and universality of mathematical truths. For Frege, all mathematical statements are analytic a priori because they are true by virtue of meaning and logical structure, independent of experience. He aimed to reduce arithmetic to logic. Kripke introduced the concept of the necessary a posteriori and showed how some truths can be necessarily true yet known only through experience. This paper compares these three thinkers and considers their strengths and limitations. Ultimately, we agree and disagree with different parts of their views. It finally reaffirms the Fregean stance that mathematical truth is necessary and a priori, with analytic statements in arithmetic and synthetic statements in geometry.

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1. Introduction:

Three stances on mathematical truths as necessary truths are examined. These stances are first placed in debate, and then an attempt is made to reach a synthesis.

First, the Fregean stance (FS), holds that the truths of mathematics are certain.[1] Whereas the physical sciences arrive at their general truths through induction, mathematical theorems rest on axioms that are self-evident and intuitive. Rosado Haddock contends that true mathematical propositions are analytic, known *a priori*, and necessary: "Traditionally, the notions of analyticity, *a priori, and necessity have been considered coextensive and also their counterparts, namely, syntheticity, a posteriori,* and contingency."[2]

Second, the Millian stance (MS) treats mathematics as an empirical science. Mathematical, especially arithmetical, truths are experiential truths like those of the natural sciences. Deductive sciences claim certainty because their propositions follow from axioms and definitions. Mill argues that the axioms are "generalization from the facts furnished to us by our senses or by our internal consciousness."[3] If axioms come from experience, then the sciences built on them are also empirical. Mill objects to classifying only mathematical truths as certain while classifying truths in other sciences as uncertain or probable. Mill points cites as evidence that children learn numbers through their senses—such as sight and touch—rather than by any other means.

Third, the Kripke stance (KS), separates epistemic concepts of *a priori* and *a posteriori* from the metaphysical concepts of necessity and contingency. Kripke argues that some necessary truths are known *a posteriori*. Kripke claims that philosophers of mathematics make the mistake of assuming that if it is in the nature of mathematical propositions that they *can* be known *a priori*, then all mathematical truths *must* be known *a priori*. But '*p* can be known *a priori*' does not imply that '*p* must be known *a priori*'. Kripke provides examples of mathematical propositions that can be known *a posteriori*.[4] Hence, Kripke denies that the nature of mathematical propositions is that they can only be known *a priori*.

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[1] Ayer, A. J. "The A Priori." In *Language, Truth, and Logic*, Penguin, 1946, pp. 37–49.

[2] Rosado Haddock, G. E. "On Analytic *A Posteriori* Statement: Are They Possible?" *Logique et Analyse*, 229 (2015): 25–33, pp. 25. https://www.jstor.org/stable/44085329, Accessed on 20 August, 2025.

[3] MILL, J. S. *A System of Logic, Ratiocinative and Inductive*, Eighth edition, Harper and Brothers Publishers,1881, New York, pp. 187.

[4] Kripke, Saul. *Naming and Necessity*. Harvard University Press, 1980, Cambridge, MA, pp. 35.

The three stances here are not stated in their historical chronological order because FS is taken to be basically the correct stance. FS is also historically grounded in Leibniz, so it predates MS and KS.

This paper investigates the nature of mathematical truth in the context of the FS, MS, and KS. In the philosophy of mathematics, the notion of 'truth' is that of mathematical states being true. The minimal notion of 'mathematical truth' for the purpose of this paper is:

MMT: A mathematical statement (equation, etc.) is true if and only if it corresponds to a mathematical fact.

Mathematical facts are of a different type than physical facts, such as the fact that the Earth is spherical. 'Mathematical truths' is used as an abbreviation for 'true mathematical statements.' Whereas the philosophy of language is focused on what makes a statement 'true' as well as what makes a statement 'false,' the philosophy of mathematics focuses almost exclusively on true mathematical propositions, ignoring false ones. At the core of the philosophy of mathematics is the nature of axioms and proofs. Axioms are always true; hence, false mathematical statements are not considered. 'Mathematical' is used in the sense that there cannot be a false mathematical statement. '7 + 5 = 13' is false because it does not correspond to a mathematical fact. But rather than saying that it is mathematically false, we simply assert that it is false. Since a false statement can be neither an axiom of mathematics nor can it be proved by the axioms and theorems, it is 'not mathematical.' Therefore, it is safe to say, since Kant, that *all*—not some or most—mathematical truths are known *a priori*.

Kant's account of the nature of mathematical truth is a precursor to MS, FS and KS and must be visited first. Kant makes two distinctions: (i) between two types of statements: analytic and synthetic[5]; (ii) between two ways of knowing: *a priori* and *a posteriori*[6]. Hence, four combinations result: (I) analytic *a priori*, (II) synthetic *a priori*, (III) synthetic *a posteriori*, (IV) analytic *a posteriori*. For Kant, (IV) is impossible. So, three possibilities remain: (I), (II), and (III). For Kant, the nature of arithmetical statements is that of category (II) synthetic statements that are known *a priori*, though some mathematical statements, such as '1=1' or some definitions, and all logical propositions are in the category of (I) analytic statements that are known *a priori*. For Kant, no mathematical statements belong to category (III) of synthetic *a posteriori*.

[5] Das, Rasvihary, *A Handbook to Kant'sCritique of Pure Reason*, Progressive Publishers, 2023, Kolkata, pp. 38.
[6] Ibid, pp. 36-37.

Kripke has provided a counterexample to Kant's denial of (IV) analytic *a posteriori*: "Water is H_2O" is analytic because the concept of water contains the concept of the chemical composition of water. However, "Water is H_2O" is known *a posteriori* because empirical observation is required to confirm its truth. Kripke conflates necessity with *a priori*. Kripke is however perfectly justified in making this conflation because Kant specifies universality and necessity as the criteria for *a priori*: "Necessity and strict universality are therefore secure indications of an *a priori* cognition, and also belong together inseparably."[7]

George Bealer, however, suggests it is better to think of the necessity–contingency distinction as a third distinction of statements rather than as a part of the *a priori–a posteriori* distinction, as Kant thought.[8] The analytic–synthetic distinction is linguistic/semantic, the *a priori–a posteriori* distinction is epistemological, and the necessary–contingent distinction is metaphysical. In contemporary terms a statement's being necessarily true means that it is true in all possible worlds. There are eight resultant combinations of the three types of distinctions when applied to statements:

a) Analytic *a priori* necessary
b) Analytic *a priori* contingent
c) Analytic *a posteriori* necessary
d) Analytic *a posteriori* contingent
e) Synthetic *a priori* necessary
f) Synthetic *a priori* contingent
g) Synthetic *a posteriori* necessary
h) Synthetic *a posteriori* contingent

For Kant, the nature of mathematical truths is that of (e) synthetic, necessary statements known *a priori.* For Frege, the nature of arithmetical truths is that of (a) analytic, necessary statements known *a priori*, but the nature of geometrical truths is that of (e) synthetic, necessary statements known *a priori*. For Mill the nature of mathematical truths is that of either (d) analytic, contingent statements known *a posteriori* or (h) synthetic, contingent statements known *a posteriori*, since Mill is not bothered much with the semantic analytic–synthetic distinction. For Kripke, it is not clear what the nature of mathematical truths is. He may agree that most

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[7] Kant, Immanuel. *Critique of Pure Reason*, translated and edited by Paul Guyer and Allen W. Wood, Cambridge University Press, 1998, B4 (pp. 137–8).
[8] Bealer, George. "The A Priori." In Ernest Sosa and John Greco (eds.), *The Blackwell Guide to Epistemology*, Blackwell Publishers, 1999, pp. 243–270, p. 243.

mathematical truths are either (a) analytic, necessary statements known *a priori* or (e) synthetic, necessary statements known *a priori because* Kripke seems to have abandoned the analytic–synthetic distinction following Quine. But Kripke denies that *all* mathematical truths are either (a) or (e) and gives counterexamples of mathematical truths that can be known *a posteriori*. Hence, some mathematical statements may be (c) analytic, necessary, and known *a posteriori* or (g) synthetic, necessary, and known *a posteriori*. But Kripke does not admit any mathematical statements as being contingent, hence ruling out (b), (d), (f), and (h).

To sum up: For Kant the nature of mathematical truths is based on (e). For Frege it is (a) for arithmetic and (e) for geometry. For Mill it is either (d) or (h). For Kripke there is no explicit commitment to the nature of mathematical truth. But Kripke definitely denies (b), (d), (f), and (h) in the realm of mathematical truths. Kripke allows all of (a), (c), (e), and (g) as types of mathematical truths. The distinction is of great use as at least (b) and (f)—that is, contingent mathematical statements are known *a priori*—are rejected by all of Kant, Frege, Mill, and Kripke as types of mathematical truths.

The plan of the rest of the paper is as follows: Section 2 is the exposition of the Millian stance on mathematical truths, including the axioms as empirical truths, and the crux is that mathematics is an inductive science. Section 3 presents the Fregean stance: Section 3.1 highlights the failure of the Millian stance to distinguish between first-level and second-level concepts, leading to the failure to recognise that numbers are second-level concepts, which are not empirical, though they are properties of empirical objects. Section 3.2 develops the Fregean stance on mathematical truths as analytic, *a priori*, and necessary. This triad captures the scope of mathematics in practice, whereas Kant and the Millian stance limit the scope of mathematics. Section 4 provides an account of the Kripke stance in which mathematical truths include at least some analytic, *a posteriori*, but necessary truths, where 'necessity' is defined in terms of 'rigid designators.' Section 4.1 is a break from the traditional acceptance of Kripke as it demonstrates that the rigidity of designators does not guarantee that statements with these designators will hold true in all possible worlds. Section 4.2 establishes that Kripke has made a fallacious jump from the fact that we may have instances of practical *a posteriori* knowledge of a mathematical statement to the conclusion that knowledge of some mathematical propositions is *a posteriori*. Knowledge of all mathematical statements by their nature remains *a priori*, though some can also be known *a posteriori*. Section 5 is an attempted synthesis in which the softened version of the Fregean stance is sustained, allowing for empirical verification of some mathematical truths and permitting *a posteriori* knowledge of mathematical truths in some branches of mathematics. But the present discussion is confined to arithmetic.

2. Millian Stance (MS):

MS is grounded in naturalism, which holds that knowledge is gained by understanding the mind, the world, and their interactions. According to Mill, all knowledge is factual and substantial, arising from experience. Humans, including their minds, are part of nature. Mill rejects the Kantian view that the mind shapes the structure of our knowledge of the universe. Kantians maintain that "if our experience could be found to take a certain form, then we would infer facts about how the world must be composed"[9], and they regard these forms as *a priori*. Mill disagrees, arguing that all knowledge, including mathematical knowledge, comes from experience or sensory observation. This is Mill's anti-*apriorism*.

Mill opposes those who claim that the "propositions of the science of numbers are merely verbal"[10] or that mathematics is "a simple transformation of language"[11]. Mill argues that "two plus one is equal to three" is not a definition but a statement about real, existing facts. For Mill, verbal propositions—defined as conditional and lacking factual content—differ from real propositions, which contain factual content. Mill insists that both mathematics and logic involve real inferences rather than mere linguistic manipulation.

The verbalist *a priori* view of numbers claims that "two plus one is equal to three" is a definition of 'three,' or a statement in which it is conventionally agreed to use the name 'three' as a sign'[12]**.** Mill argues that this "artful manipulation of language is so contrary to common sense"[13]. Mill claims that in every step of an arithmetical calculation, there is real induction or real inference of facts from facts. Every number represents something; numbers are not abstract in nature. For example, 'three' means three books or 'five' means five cats, and so on. He describes propositions concerning numbers as "propositions concerning all things," which can be broken into parts, and each part can be counted as a number, and it can be known through our experience[14].

[9] Macleod, Christopher. "John Stuart Mill." *The Stanford Encyclopedia of Philosophy* d N. Zalta (ed.). https://plato.stanford.edu/archives/sum2020/entries/mill/ Accessed on 25 August, 2025.
[10] MILL, 1881 pp. 188.
[11] Ibid.
[12] Ibid.
[13] Ibid.
[14] Ibid., pp. 189

In contrast to Frege, who assigns numbers to concepts[15], Mill assigns numbers to aggregates—the totality of things. For Mill, "a numeral represents a cardinality property of some things"[16]. The number 2, for instance, is a cardinal property of an aggregate and refers to a set containing exactly two items. Whereas for Frege, 2 would refer to a set of all pairs. These cardinal properties are non-distributive: if five things are windows, each one of the five is a window, but if four things are four, each of the four is not four. MS supports empiricism because aggregates are observable pluralities.[17] 'Five plus one is equal to six' means that five apples and one apple together make six apples. Yet these two classes are not absolutely identical. Their denotations are the same, but their connotations differ. For example, six apples in three bags are not the same as six apples in one bag, since they make different impressions on our senses[18]. And the two collections of apples are situated in different places. If we consider the absolute identity between the two collections, we shall not be able to get any new knowledge.

According to Linnebo, Mill views arithmetic as the study of pluralities and their non-distributive cardinal properties.[19] Linnebo furthermore claims that for MS mathematical laws are known experimentally and confirmed inductively by their examples. Mill also holds that mathematical laws are known through experience and confirmed inductively. For instance, '2 + 2 = 4' can be represented as two disjoint pairs of things; those pairs, which have no element in common, can be combined to make a quadruple of things.[20] Two non-overlapping pairs, for example, one with two bananas and the other with two oranges, together will represent the number 4.

Mill concludes that mathematical truth is an inductive truth, knowable only through experience. While some might say that "six is five plus one" is a definition, Mill sees it instead as an arithmetical theorem stating that "a collection of objects exists."[21] The truths of mathematics are approximate or hypothetical because mathematics is an inductive science. Hence, the necessity of mathematical truths is not assured.

3. Fregean Stance (FS):

[15] Linnebo, Ø. *Philosophy of Mathematics*. Princeton University Press, N.J. 2017, Princeton, pp. 26.
[16] Ibid., pp. 90.
[17] Ibid. pp. 90.
[18] MILL, 1881, pp. 190.
[19] Linnebo, 2017, pp. 90.
[20] Ibid.
[21] Ibid.

FS holds that mathematical truths are necessary truths derived from axioms, which are also necessary and not empirical, whereas in the Millian stance they are hypothetical or approximate and empirical.

3.1 FS Critique of MS:

3.1.1 Frege's Objections to Mill's Theory of Mathematical Truth:

According to Mill, the definitions '2 = 1 + 1' and '3 = 2 + 1' cannot be constructed unless the facts to which these definitions refer are observed. Frege accepts that the number '3' cannot be defined as '2 + 1' until a sense is given to '2 + 1'. However, this does not imply that one must always see a collection or separation of objects to which it refers. Otherwise, explaining the number 0 would be problematic since no one has ever touched or seen 0 apples[22].

Mill holds that every number corresponds to physical objects or facts. Frege, however, argues that such correspondence is possible only for smaller numbers, not for very large numbers like 100000000000. Therefore, MS cannot account for all numbers, because some numbers, like 0, have no objects associated with them, while others involve quantities too large to be perceived or counted.[23] The difficulty becomes greater with infinite numbers, which MS would have to account for, since infinite numbers cannot in principle be perceived or counted.

Frege claims:

> "MILL understands the symbol '+' […] to express the relation between the parts of a physical body or of a heap and the whole body or heap; but such is not the sense of the symbol. […] MILL always confuses all the applications that can be made of an arithmetical proposition, which often are physical and do presuppose observed facts, with the pure mathematical proposition itself."[24]

Frege maintains that there is a distinction between the meaning of a representation and its application. For instance, "the process of heaping up" constitutes merely one application of the symbol '+'; it does not express its meaning. According to Frege, Mill failed to recognize this

[22] Frege, G. *The Foundations of Arithmetic,* Translated by J. L Austin. Harper and Brothers, 1960, , New York, §8 pp. 11.
[23] Ibid.
[24] Ibid., §9, pp. 13.

distinction. Rejecting any physical relationship as corresponding to addition, Frege infers that "the general law of addition cannot be the law of nature."[25]

In Frege's view, a number is a concept. The term 'concept' is employed in two distinct senses: a first-level concept and a second-level concept. Whereas, for Mill, a number is an observed aggregate of physical objects. On the MS account, mathematical truths would be contingent, and mathematics itself would be limited, since a finite being can observe only a finite number of physical facts. Frege rejects MS to preserve the necessary and universal character of arithmetic.

Consider an example: there are six chocolates; here, 'chocolates' is a first-order concept, while 'six' is a second-order concept, namely, a number ascribed to the first-order concept 'chocolates.' First-order concepts such as 'chocolates' refer to spatiotemporal physical things, whereas second-order concepts like the number six are non-spatiotemporal things. This sentence may be formally represented as 6x(Chocolate(x))[26]. It must be noted, however, that many hold numbers to be objects rather than second-order concepts.

3.1.2 Ayer's Critique of MS:

For MS, mathematical truths are conditional or experimental rather than *a priori* or necessary because mathematics is regarded as an empirical science, akin to other branches of science. This is captured succinctly by Mill:

> "What we have now asserted, however, cannot be received as universally true of Deductive or Demonstrative Sciences, until verified by being applied to the most remarkable of all those sciences, that of numbers; the theory of the calculus; arithmetic and algebra. It is harder to believe of the doctrines of this science than of any other, either that they are not truths *a priori*, but experimental truths, or that their peculiar certainty is owing to their being not absolute but only conditional truths."[27]

A. J. Ayer challenges Mill's claim that mathematical propositions are akin to those propositions of other empirical sciences[28]. Ayer argues that mathematical truths differ fundamentally from

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[25] Ibid. §9, pp. 13-14.
[26] Linnebo. 2017, pp. 26.
[27] MILL, 1881, pp. 188.
[28] Ayer, A. J. "The A Priori", From *Philosophy of Mathematics: Selected Readings*, Edited by Paul Benacerraf and Hilary Putnam. New Jersey, Prentice-Hall, 1964, Princeton, pp. 289–301; p. 291.

empirical propositions because they are certain and universal, and their validity is independent of empirical verification. Even when discovered through inductive generalization, mathematical truths are recognized as necessary rather than probable, and thus "they hold good for every conceivable instance."[29]

Ayer illustrates this with an example. According to Mill, '3 × 5 = 15' is an empirical generalization. If, in a particular case, counting yields '3 × 5 = 12', would this refute '3 × 5 = 15'? If mathematical propositions were inductive generalizations, the answer would indeed be "yes." However, if mathematical truths are not empirical generalizations—as maintained by FS and supported by Ayer—then the proper conclusion would be that an error occurred in identifying a group of 3 or a group of 5, or in counting. Such an observation does not undermine the mathematical truth '3 × 5 = 15.' Because these truths are analytic and *a priori*[30], they cannot be refuted by empirical counterexamples. Consequently, Ayer asserts, "Mill was wrong in supposing that a situation could be raised that would overthrow a mathematical truth,"[31] and adds that such a situation could never occur.

To appreciate Ayer's argument fully, it is important to clarify his notion of analyticity. He defines an analytic proposition as one whose "validity depends solely on the definitions of the symbols it contains"[32] and a synthetic proposition as one whose validity "is determined by the facts of experience."[33] Mathematical propositions are analytic because their validity can be established by analyzing their constituent symbols or because mathematical expressions are synonymous with one another. For example, '4 + 2' is synonymous with the number 6, just as a *gynaecologist* is synonymous with a woman's doctor. The validity of the number 6 can be demonstrated by analyzing its components—'4' and '2'—connected by the '+' sign. In this way, the proposition "4 + 2 = 6" is both analytic and necessary.

Ayer describes the denial of an analytic proposition as "self-stultifying," since such denial is self-contradictory. For this reason, he holds that mathematical analytic propositions are necessary and that no situation could be raised that would overthrow a mathematical truth. He further claims that "the same explanation holds good for every other *a priori* truth"[34].

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[29] Ibid., pp. 292
[30] Ibid.
[31] Ibid., pp. 293
[32] Ayer, J, A, *Language, Truth, and Logic*, Dover Press, 1946, New York, pp-29.
[33] Ibid.
[34] Ibid, pp-85.

3.1.3 Carl G. Hempel's criticism of MS:

Hempel argues in a similar fashion to Ayer. He argues that in empirical sciences, such as the law of gravitation, one can identify conditions under which the law fails. By contrast, he questions whether any circumstance could show that "2 + 2 = 4" is false. He illustrates this with a case: a researcher places two microbes first, then two more, and counts them. If the count gives five, Hempel asks, "Will it be the case where we can disconfirm the mathematical proposition like '2 + 2 = 4'?" He answers, "Clearly not," since the error lies in counting, not in the proposition itself. As he explains, the expressions '2 + 2' and '4' refer to the same number because the symbols ('2', '2', '4', and '+') are defined in such a way that the identity holds by meaning alone[35]. This makes the statement analytic rather than empirical. For this reason, mathematics should be understood as different from empirical science.

In short, Hempel maintains that mathematical truths are "true by definition,"[36] analytic, and independent of empirical validation. Their truth rests on the meanings of terms. Finally, he adds that no empirical test can establish or disconfirm logical or mathematical propositions, since they are analytic or *a priori* truths.

It is to be noted that a proposition is *true by definition* when its validity follows solely from the meanings of its terms, such that its denial entails a contradiction (e.g., all bachelors are unmarried men). A proposition is *true by axiom* when it is accepted without proof as a foundational assumption within a formal system (e.g., through any two distinct points, there is exactly one straight line in Euclidean geometry). The former rests on linguistic meaning, whereas the latter rests on stipulation within a deductive structure. The distinction between Frege's and Ayer's (along with Hempel) positions concerns the grounding of mathematical truth: Frege holds that mathematical propositions are true insofar as they are derived from axioms, whereas Ayer maintains that their truth rests on their being true by definition. Although the approaches differ, the underlying aim remains the same, namely, to conceive of mathematical truths as analytic and known *a priori*.

3.2 Frege's Stance (FS) Regarding Mathematical Truth:

FS asserts that arithmetical truths are necessary, known *a priori*, and analytical. According to him, the truths of mathematical propositions sometimes rely on several logical definitions that

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[35] Hempel, G, C, "On the Nature of Mathematical Truth", From *Philosophy of Mathematics: Selected Readings*, Edited by Paul Benacerraf and Hilary Putnam, New Jersey, Prentice-Hall, 1964, Princeton, pp.368.
[36] Ibid, pp-368

are not expressed explicitly, much like a plant 'contained in'[37] a seed. These truths are also 'contained in' certain logical definitions or axioms. In other words, since every proposition of arithmetic derives from laws of logic, it can be claimed that the arithmetical truths are analytic and known *a priori*. He says, "I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgments and consequently *a priori.*"[38]

Furthermore, Frege claims that it is possible to get new knowledge from these analytic propositions. Frege has given a proof of a proposition to show how it is possible to get new knowledge from an analytic proposition: "If the relation of every member of a series to its successor is one or many-one, and if *m* and *y* in that series after *x*, then either *y* comes in that series before *m*, or it coincides with *m*, or it follows after *m*."[39]

Imagine a series (like a line of things: $x \rightarrow m \rightarrow y$). Each thing is followed by another in a clear order.

That order could be:

> One-to-one, that is, each thing has only one next thing, or many-to-one, that is, many things can lead to one next thing. Now, if both *m* and *y* come after *x* in that line, then
>
> Either *y* comes before *m*, or
>
> *y* is the same as *m*, or
>
> *y* comes after m.

These are the only three possibilities. Nothing else is possible in this series. One might contend that if mathematical propositions are taken to be analytic *a priori*, they would amount to tautological statements, disclosing no new knowledge beyond what is already contained in the concepts themselves. But this is not a true allegation because one can understand how numbers are ordered, which is new information. So, one learns something new or extends their knowledge; nevertheless, mathematical propositions are regarded as analytic *a priori*. Frege claims that one obtained this result just by logic—not by looking at the real world. It demonstrates how an analytical, known *a priori*, and necessary proposition—since it holds true

[37] Frege, G. *The Foundations of Arithmetic,* Translated by J. L Austin. Harper and Brothers, 1960, New York, §87, pp. 100.
[38] Ibid, pp-99.
[39] Ibid, §90, pp. 103.

for all possible worlds—can also aid in knowledge expansion by supplying new information. On this basis, he criticizes Kant, who maintains that mathematical truths are synthetic *a priori*. Furthermore, the Fregean stance presented here is for arithmetical truth, and 'mathematical truth' is loosely used for arithmetical truth.

4. Kripke's Stance (KS) on Necessity and Apriority

Saul Kripke's idea of the necessary *a posteriori* revolutionized 20th-century metaphysics and philosophy of language by challenging the long-standing assumption that all necessary truths are knowable *a priori* and all contingent truths are knowable *a posteriori*. In his seminal *Naming and Necessity* (1980), Kripke argued that there are truths that are both necessarily true and yet only knowable through empirical means—these are the necessary *a posteriori* truths.

To understand Kripke's contribution, it is important to first separate two types of distinctions in philosophy: the epistemological distinction between *a priori* and *a posteriori* and the metaphysical distinction between necessity and contingency. Traditionally, philosophers such as Hume and Kant thought that a proposition was either necessary and knowable *a priori* (e.g., all bachelors are unmarried) or contingent and known *a posteriori* (e.g., the sky is blue). Kripke questioned this neat division.

One of Kripke's core arguments involved the nature of names and rigid designation. According to him, proper names (like Aristotle or H_2O) are rigid designators—they refer to the same object in all possible worlds where that object exists.

A proper name is a rigid designator if it refers to the same object in the actual world and in all possible worlds where that object exists, and nothing else can bear that name. For example,

(1) 'Hesperus = Phosphorus'

is necessarily true because both names are rigid designators that refer to the same object in all possible worlds. By contrast, definite descriptions are not rigid designators.

In

(2) 'Hesperus = the brightest non-lunar object in the evening sky.'

Though the reference of the name 'Hesperus'[40] as a rigid designator is the same in all possible worlds, the definite description "the brightest non-lunar object in the evening sky" does not refer to the same object in every possible world. In some possible world, it may turn out that Mars is the brightest non-lunar object in the evening sky because Hesperus is blotted out by some cosmic dust.[41] Hence, (2) is not necessarily true, since (2) is false in at least one possible world, even though it is true in many possible worlds.

Kripke also claims that rigid designators can occur in *a posteriori* necessary truths.[42] "Water is H_2O" as a theoretical identity[43] is necessarily true because both terms designate the same substance in all possible worlds, but this identity is *a posteriori* because it was discovered through scientific empirical investigation.[44] The identity statement "Water is H_2O"[45] was once discovered through empirical chemistry and found to be necessarily true. That is, in any possible world where water exists, it is composed of H_2O. Yet we did not know this *a priori*; it required empirical investigation. Hence, it is a necessary truth known *a posteriori*. The same is the case with 'Hesperus is Phosphorus.' Ancient astronomers observed Hesperus (the evening star) and Phosphorus (the morning star) without knowing they were both Venus. The identity "Hesperus is Phosphorus" is necessarily true because Venus is identical to itself in all possible worlds. But the discovery was empirical—it came through observation, not pure reasoning. This again illustrates a necessary *a posteriori* truth.

Kripke's insight upended the views of logical positivists like A.J. Ayer, who held that necessity and *apriority* were coextensive. Kripke demonstrated that modal and epistemic properties are distinct; just because something is necessarily true does not mean we can know it without experience, and just because something is contingently true does not mean we cannot know it through logic or reason.

This distinction also has important implications for metaphysics. For instance, Kripke used the necessary *a posteriori* to argue for essentialism: the idea that certain properties are essential to objects. If water is necessarily H_2O, then anything not composed of H_2O is not water—even if

[40] Kripke, *Naming and Necessity*, published by Harvard University Press, 1980, Cambridge, MA, pp. 57.
[41] Kripke, Saul. *Naming and Necessity*.
[42] LaPorte, Joseph. "Rigid Designators." *The Stanford Encyclopedia of Philosophy* (2022 Winter Edition). 2022. Available at https://plato.stanford.edu/entries/rigid-designators/. Accessed on 26 June, 2025.
[43] Ibid., pp. 116.
[44] Ibid., pp. 128.
[45] Kripke, *Naming and Necessity*, pp. 148.

it looks and behaves like water. Thus, the chemical composition is essential to the identity of water.

According to Kripke, proper understanding of rigidity may open the chance of contingent *a priori* statements, that is, Kripke considers that a contingent statement can be known *a priori*:

> The standard meter stick, which has been used to define the length of one meter. Someone decreed that the expression 'one meter' is to be used for the unit of length of that stick, perhaps at a certain time in the stick's existence. The stick, S, might not have been the length that it was at the time 'one meter' was coined: had it been heated or cooled it might have been longer or shorter. As things are, it was not heated or cooled. So, the sentence 'One meter = the length of S (at time t0)' is true in the actual world but it is false in some possible worlds: it is contingently true. Yet S has been used to define 'one meter,' in the sense that it fixes the reference, so, the definers of 'one meter' know *a priori* the truth of the sentence.[46]

In sum, Kripke's notion of the necessary *a posteriori* or contingent *a priori* fundamentally reshaped philosophical thinking about meaning, identity, and metaphysical necessity.

Now let us turn to how Kripke's breakthrough may apply to the nature of mathematical truths.

Philosophers often describe mathematical truths as belonging to the realm of *a priori* knowledge. To say that something is *a priori* means that it can be known independently of experience, through pure reasoning or logical calculation. For example, one can know that "2 + 2 = 4" without looking at the world or performing an experiment. This has led many to assume that all mathematical truths *must* be known in this way. However, this assumption is too strong. There are good reasons to believe that some mathematical truths, while capable of being known *a priori*, are sometimes actually known *a posteriori*, that is, they are known through experience or observation.

The confusion often comes from changing *can* to *must*. Just because a truth *can* be known by pure reasoning does not mean that it *must* always be known that way. Human beings have limited time, energy, and reasoning ability. While in principle we could calculate or prove

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[46] LaPorte, Joseph, "Rigid Designators", The Stanford Encyclopaedia of Philosophy (Winter Edition), Edward N. Zalta & Uri Nodelman (eds.), 2022, https://plato.stanford.edu/archives/win2022/entries/rigid-designators/, Accessed on 26 August, 2025.

certain mathematical facts ourselves, in practice we often rely on tools, instruments, and even physical processes to arrive at these truths. This introduces the possibility of knowing mathematics on an *a posteriori* basis.[47]

Take the example of prime numbers. Whether a particular number is prime or not is, by nature, a mathematical fact. In principle, it can be shown through a logical proof that a number is prime or composite. This makes it an *a priori* truth. However, suppose a person uses a computer to check whether a large number is prime. The computer runs an algorithm, carries out the calculation, and provides an answer. The person then believes the computer's result without doing the calculation themselves. In this case, the knowledge is not gained purely through reason alone, but through trust in the functioning of the machine, its physical construction, and the principles of physics that govern its operation. This kind of knowledge is clearly *a posteriori*, since it depends on empirical evidence about how the machine works and how reliable it is.[48]

This shows that the distinction between *a priori* and *a posteriori* is not always about the truths themselves, but about the *methods* by which they are known. A mathematical truth may belong to the class of statements that *can* be known *a priori*, yet, for practical reasons, it is sometimes accessed only *a posteriori*. Thus, it is important not to confuse *can* with *must*.

Hence, mathematical truths are not always tied to one single mode of knowledge. While they are in principle *a priori*, human beings often approach them through empirical means.

4.1 Criticism against Kripke's Notion of Rigidity

Various arguments have been raised against Kripke's notion of rigidity by specialists in the philosophy of language. Some critics believe that Kripke's theory relies heavily on 'trans-world'[49] identity—that we can speak meaningfully about the same individual existing in different possible worlds. But some philosophers challenge the metaphysical assumptions underlying this. David Lewis, for instance, argued for a counterpart[50] theory, where individuals in other possible worlds are not literally the same person but rather 'counterparts'—similar but

[47] Kripke, *Naming and Necessity*, pp. 35.
[48] Ibid., pp. 35.
[49] Lewis, D, *On the Plurality of Worlds*, Oxford Blackwell, 1986, New York pp. 192.
[50] Ibid, pp.194.

not identical beings. This casts doubt on the idea that names can rigidly designate across worlds because there may not be a strict cross-world identity to preserve.

Another challenge is whether rigid designation applies equally well to natural kind terms like 'water.' Some critics point out that our scientific understanding of substances evolves, and the very criteria we use to determine identity may shift. Suppose that in another possible world, the clear, drinkable liquid we call 'water' is composed of XYZ, not H_2O. Kripke would insist that XYZ is not really water. But critics argue that this is a stipulative and perhaps overly rigid use of language that ignores actual linguistic and conceptual practices, where "water" might be understood functionally or phenomenologically, not just by chemical composition. According to Gibbard "it makes no sense to say that an object in some other world is identical to an object in the actual world. This is counterintuitive."[51]

Because Kripke's concept of rigidity[52] has some drawbacks, some critics reject it in this way. Rigidity, for instance, ensures that the same assertion, "water is H_2O," must be true in the real world. However, just because something is true in this world does not mean it must be true in other possible worlds; in other words, rigidity does not guarantee that an identical assertion holds universally across worlds.

4.2 Criticism of Kripke's Example of some mathematical statements being *a posteriori*

Let us consider the propositions:

(i) '2+2 = 4'

(i) is ambiguous in that it could either be an ordinary statement that an ordinary person wants to know for practical purposes, or it could be a statement that someone wants to know mathematically. Why is 2+2=4? Can it be proven? Let us keep the statement the same and shift to the ambiguity of the word "know." The claim that

[51] LaPorte, Joseph,"Rigid Designators", The Stanford Encyclopaedia of Philosophy (Winter Edition), Edward N. Zalta & Uri Nodelman (eds.), 2022, https://plato.stanford.edu/archives/win2022/entries/rigid-designators/, Accessed on 26 August, 2025.

[52] Kripke, *Naming and Necessity*, pp. 48.

(K*i*) "I know '2 + 2 = 4.'"

is ambiguous, as it could mean know for practical reasons or *know mathematically*. Let us then disambiguate (K*i*) as two different types of knowing:

(Ki_p) "I practically know '2 + 2 = 4.'" and

(Ki_m) "I mathematically know '2 + 2 = 4.'"

Kripke has conflated (Ki_p) and (Ki_m). The example he gives—prime numbers through computer generation—is an example of practical knowledge, not mathematical knowledge.

Let us consider the number 1733. I do not immediately know whether it is prime or not. So, I ask the computer. It immediately tells me that it is prime. Now, since I trust the computer as mentioned above, I have acquired practical knowledge of

(3) '1733 is a prime number.'

So, I can now claim:

($K\ (3)_p$) "I practically know '1733 is a prime number.'"

And the knowledge in ($K\ (3)_p$) is *a posteriori* knowledge. But I do not know it mathematically.

~ ($K\ (3)_m$) "I do not mathematically know '1733 is a prime number.'"

Hence, practical *a posteriori* knowledge of a mathematical statement does not imply *a posteriori* mathematical knowledge of the statement. Suppose I am not satisfied with this and work this out on pencil and paper. 1733 is odd, so I it will not divide without a remainder by the first prime number 2, so I move to the second prime number and divide 1733 by 3, and find out that 3 does not divide 1733 but is 644 and a remainder of 1, then I try to divide it by 5 and find that it is 346 with a remainder of 3, then I try to divide it by 7 and find that it is 247 with a remainder of 4, then I try to divide it by 11 and find that it is 157 with a remainder of 6, then I divide it by 13 and find that it is 133 with a remainder of 4, then I divide it by 17 and find that it is 101 with a remainder of 16, then I divide it by 19 and find that it is 91 with a remainder of 4, then I divide it by 23 and find that it is 75 with a remainder of 8, then I divide it by 29 and find that it is 59 with a remainder of 22, then I divide it by 31 and find it is 55 with a remainder of 28, then I divide it by 37 and find that it is 46 with a remainder of 31, then I divide it by 41 and find that it is 42 with a remainder of 11. Clearly, I do not need to go any further. I have divided 1733 by all the prime numbers from 3 to 41. The next prime number is 43 and if 1733 were divisible by it without a remainder, it would have to be by a number lower than 41 since 43 times 41 is 1763, which is greater than 1733. And I have already tried all the prime numbers below 41. And it cannot be divisible by any even number because it is an odd number. And it cannot be divisible by any odd number below 41 because if it were, let us say, divisible by 21, then it would also be divisible by the prime factors of 21—namely, 7 and 3. But we have already determined that it is not divisible by these factors.

Now, with a little patience and hard work, I can claim to know mathematically that '1733 is a prime number.' Kripke's example is of practical *a posteriori* knowledge of a mathematical proposition; it is not an example of mathematical *a posteriori* knowledge of a mathematical proposition. Kripke therefore presents his example by committing the fallacy of equivocation. Kripke even comes close to saying that the process I have gone through in the previous paragraph is *a posteriori*, since it follows from a calculating process I have undertaken. I do not mathematically know '1733 is a prime number' at the beginning of the process but only at the end of the process. Well, this would be a clear misuse of the meaning of the word '*a posteriori.'* '*A posteriori*' means prior to empirical or perceptual experience. It does not even mean 'prior to experience.' The process of calculation I have gone through is independent of perceptual experience; every step is *a priori*. Kripke would not deny that proofs are *a priori,* and surely most mathematical knowledge is posterior to, not prior to, proofs.

However, there may indeed be examples of mathematical propositions that can be known *a posteriori* and perhaps only known *a posteriori*. There are so many branches of mathematics. So, with proper research, we may be able to find such an example of mathematical knowledge. But this would only commit us to denying that all mathematical propositions are *a priori* with respect to mathematical knowledge, but we can still maintain that the nature of mathematical truth is such that the mathematical knowledge of it is almost always *a priori*.

5. Conclusion

Mill appears to conflate epistemology with metaphysics: that is, he assumes that if a proposition is known *a posteriori*, then its content must also be *a posteriori* in nature. This leads to problematic implications. For example, the mathematical proposition "7 + 5 = 12" may be learned through empirical means (thus known *a posteriori*), but this does not entail that the proposition itself is empirically grounded or *a posteriori* in nature. The truth of such a proposition remains necessary and analytic, even if its acquisition is empirical. Mill's failure to distinguish between the epistemological route to knowledge and the proposition's metaphysical status is thus a significant flaw.

Kripke argues that some mathematical truths are both necessary and known *a posteriori*. For instance, he cites conjectures such as Goldbach's Conjecture as examples that, if proven, might represent necessary truths not known *a priori*. However, this view raises further complications. The distinction between *a priori* and *a posteriori* knowledge is not necessarily mutually exclusive in all cases. One could argue that certain mathematical truths, while not *a priori* in the traditional sense, also fail to be *a posteriori*, insofar as they are not verifiable through empirical means. Take Goldbach's Conjecture again: it is neither currently known *a priori* nor can it be confirmed *a posteriori*, as it cannot be verified through empirical observation. Thus, labelling it as an *a posteriori* truth might be misleading. The deeper question then arises: What is the ontological and epistemological status of such mathematical truths? And further, are such truths problematic or undecidable within Kripke's framework?

Hence, we can carry forward from Mill the aspect that ordinary mathematical truths such as "7 + 5 = 12" can be empirically confirmed through perception as well as counting. But we reject Mill's claim that all mathematical truths are contingent and *a posteriori*. We carry forward from Kripke his assertion that some mathematical truths may be known *a posteriori* by some persons and even the claim that there may be some mathematical truths that can never be known *a priori*, but we deny that this *a posteriori* knowledge counts as mathematical knowledge; it is practical knowledge instead. Hence, we conclude, holding on to the Fregean stance, that the nature of mathematical truth, for the purposes of mathematical knowledge, is that they are necessary truths, known mathematically *a priori* and in the case of arithmetic, they are analytic statements, while in geometry, they are synthetic statements, which are nonetheless necessary truths known *a priori*. Nonetheless, in a softened Fregean stance, we allow for the possibility of there being some *a posteriori* necessary mathematical truth in some branch of mathematics that further research will discover, but we still maintain a revisionary Fregean stance as the best of all the available stances in the philosophy of mathematics on the nature of mathematical truth.

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